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The Rotation of Hyperion [and Discussion]

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The rotation of Hyperion

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For almost the entire range of dimensions of Hyperion allowed by uncertainties in the observations, the satellite cannot librate stably about a rotation rate which is synchronous with its orbital mean motion. Rather, the large gravitational torques on the asymmetric satellite coupled with the large eccentricity forced by the orbital resonance with Titan cause Hyperion to tumble in a random manner. Large changes in orientation of body axes relative to inertial space and in the instantaneous spin rate occur on timescales of the order of the orbit period. Numerical evaluation of the exponential divergence of nearby trajectories in the phase space of the motion verifies that the tumbling is truly chaotic. This newly defined state of chaotic rotation for Hyperion is likely to be the only example of confined, continuously observable chaotic motion in the Solar System.

1. Introduction

A typical rotation history for a natural satellite is a retardation of the spin by tidal friction toward a value synchronous with its orbital mean motion. Many satellites have reached this final state of synchronous rotation including the Moon, the two satellites of Mars, the Galilean satellites of Jupiter, most of the satellites of Saturn, Neptune's satellite Triton and probably most of the satellites of Uranus. As Hyperion is reasonably far from Saturn and is not too large, the timescale for the tidal retardation of its spin is comparable with the age of the Solar System (Peale 1977). However, Iapetus is even further away and is known to be rotating synchronously with its orbital motion. So it is most likely that Hyperion's spin has also tidally evolved to a value comparable with its orbital mean motion.

For Hyperion's reasonably large orbital eccentricity (0.1) there is a chance that it might have been trapped in a non-synchronous spin–orbit resonance like that of Mercury where the spin angular velocity is 1.5 times that of the orbital mean motion (Goldreich & Peale 1966; Peale 1978). This possibility depended on Hyperion's not being axially symmetric, but the deviation from symmetry could not be too large since averaging over high-frequency terms, which isolates the libration term in the equation of motion, requires that the libration of the spin about the resonant angular velocity has a very long period. However, images of Hyperion by the Voyager spacecraft (Smith *et al.* 1982) revealed a shape so asymmetric that earlier analyses of satellite and planetary spins involving averaging over high-frequency terms are not applicable. The possible spin state where Hyperion is rotating stably at an angular velocity which is 1.5 times its orbital angular velocity as suggested by Peale (1978) does not even exist. It is most likely that Hyperion will always tumble in a chaotic way with the magnitude and direction of its spin vector exhibiting large variations on timescales comparable with the orbital period (Wisdom *et al.* 1984).

The route to this conclusion is developed below by first considering in §2 a restricted problem where the spin axis is normal to the orbit plane. With this axis orientation, both chaotic rotation and stable libration about spin angular velocities which are commensurate with the orbital mean motion are demonstrated. But we show in §3 that the orientation of the spin axis normal to the orbit plane is unstable for all chaotic rotation and for libration about synchronous as well as half synchronous rotation. A discussion follows in §4 where the likelihood of finding Hyperion tumbling chaotically at the present time is supported.

2. Chaotic rotation with zero obliquity

The chaotic rotation is best understood by recalling some of the details of the analysis of spin–orbit coupling and showing where that analysis fails in the case of Hyperion. We first consider the case where the spin axis of a planet or of a satellite remains perpendicular to its orbit plane (zero obliquity). In the absence of a tidal torque, the equation of motion for the spin is (Goldreich & Peale 1966)

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{3}{2}\frac{Gm}{r^3}\frac{(B-A)}{C}\sin 2(\theta-f) = -\tfrac{1}{2}\omega_0^2 \sum_{k=-\infty}^{\infty} H(\tfrac{1}{2}k, e)\sin(2\theta - kM), \tag{1}$$

where the spin axis coincides with the axis of maximum moment of inertia, θ is the angle between the axis of minimum moment of inertia (in the orbit plane) and a line from the primary to the orbit periapse, f is the true anomaly, $A < B < C$ are the principal moments of inertia, m is the primary mass, r is the primary–satellite separation and $\omega_0^2 = 3(B-A)\,n^2/C$ with $n = (Gm/a^3)^{\frac{1}{2}}$ the orbital mean motion. The series expansion in the mean anomaly M follows from the periodicity of $(a^3/r^3)\sin 2f$ and $(a^2/r^3)\cos 2f$. $H(\frac{1}{2}k, e)$ are series in e each with factor $e^{2(\frac{1}{2}k-1)}$ which are tabulated by Cayley (1859) and, for a few values of k, by Goldreich & Peale (1966). A spin–orbit resonance occurs when $\dot{\theta} = pn + \dot{\gamma}$ for some half integer p, where $\dot{\gamma}/n \ll 1$. Since $n = \dot{M}$, the term in the infinite series with $k = 2p$ is slowly varying. With $\theta = pM + \gamma$, γ is the angle between the long axis of the satellite and the direction to the primary when the satellite is at periapse. With this substitution (1) becomes

$$\frac{\mathrm{d}^2\gamma}{\mathrm{d}t^2} + \tfrac{1}{2}\omega_0^2 H(p, e)\sin 2\gamma = -\sum_{k=-\infty,\, k\neq 2p}^{\infty} H(\tfrac{1}{2}k, e)\sin[(2p-k)\,M + 2\gamma], \tag{2}$$

and $\omega_0\sqrt{H(p,e)}$ is seen to be the frequency of small oscillations in γ.

Every term on the right side of (2) contains M and therefore oscillates at a high frequency compared with the term containing only γ. If $(B-A)/C \ll 1$, γ does not change much during an orbit period and we can average the right side over this period while holding γ constant. The resulting averaged equation of motion is (2) with the right side equal to 0. This averaging procedure is not valid if $(B-A)\,H(p,e)/C$ is so large that γ varies substantially over an orbit period. For the application of the theory to a nearly spherical Mercury by Goldreich & Peale (1966), the averaging procedure is valid and the resulting pendulum equation in (2) demonstrated the stable libration of the spin angular velocity about $\dot{\theta} = pn$ for several values of p in addition to the observed resonance at $p = \frac{3}{2}$.

Without the right side, (2) is integrable but is valid only near a resonance. The first integral

$$\tfrac{1}{2}\dot{\gamma}^2 - \tfrac{1}{4}\omega_0^2 H(p,e)\cos 2\gamma = E \tag{3}$$

can be used to make a phase plane plot of $\dot{\gamma}$ against γ which shows the character of the motion.

We can map all possible motions by choosing a range of values of E. For $-\frac{1}{4}\omega_0^2 H(p,e) < E < \frac{1}{4}\omega_0^2 H(p,e) = E_c$ the motion is a libration, and a unique trajectory in the phase plane results for each value of E in this range. For $E > E_c$ the motion is either a positive or negative circulation, and E determines either of two trajectories in phase differing by the sign of $\dot{\gamma}$. For $E = E_c$ the trajectory is the infinite period separatrix separating libration from rotation. In figure 1 we have added pn to $\dot{\gamma}$ and plotted representative phase trajectories

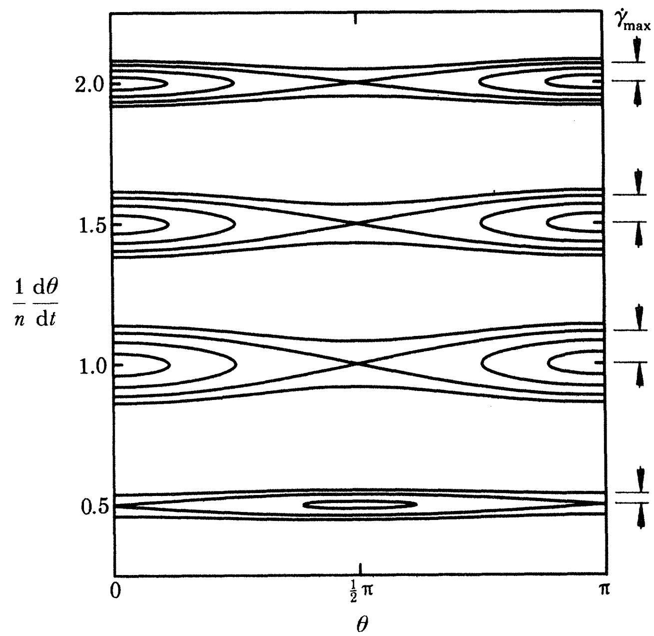

FIGURE 1. Phase trajectories near spin–orbit resonances for averaged equations of motion.

about several resonances. As 2γ appears in the cosine, we have restricted the phase plane plot to $0 \leqslant \gamma \leqslant \pi$ and replaced γ by θ on the abscissa with the understanding that this is that part of θ that exceeds an integral multiple of π when the satellite is at periapse ($M = 0$). Large values of $\omega_0 = 0.12n$ and $e = 0.2$ were used to better illustrate the trajectories in figure 1, although these are outside the range of validity of the averaging procedure.

If librations are completely damped, $\dot{\gamma} \equiv 0$, ($\dot{\theta} = pn$) and the phase plane plot reduces to a single point which we may place at either 0 or π. If $E < E_c$ libration curves fall inside the separatrix as closed curves. (Positions 0 and π are equivalent and the curves are closed if we wrap figure 1 around a cylinder.) For $E > E_c$, the curves stretch across the diagram representing positive or negative relative circulation. The curves are still closed on the cylinder, but θ changes without bound. The intersection of the separatrix with the line $\dot{\theta}/n = p(\dot{\gamma} = 0)$ corresponds to the unstable pendulum equilibrium, and $|\dot{\gamma}|$ reaches a maximum value of $|\dot{\gamma}_{max}|$ when $\gamma = 0$ or π on the curve. This value of $|\dot{\gamma}_{max}|$ on the separatrix is the halfwidth of the resonance since if $|\dot{\gamma}|$ were any larger at this point, $\dot{\theta} - pn = \dot{\gamma}$ would circulate instead of librate. For Mercury with $(B-A)/C = 10^{-4}$ and $p = 1.5$, $|\dot{\gamma}_{max}| = 0.014n$.

As plotted in figure 1, each phase diagram represents trajectories corresponding to exact integrals of an approximate equation, and any trajectory is traversed smoothly for given initial values of γ and $\dot{\gamma}$. However, the resonance equations are only valid when $\dot{\gamma}$ is small ($\dot{\theta}$ near

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pn) so we cannot plot points far away from a resonance. However, the form of the phase plane plot suggests the following procedure: we can check the validity of the approximation and plot rotation trajectories between resonances by integrating the exact equation of motion (1) numerically (equivalent to keeping the high frequency terms in (2)) and plotting the values of $\dot{\theta}$ and θ each time the satellite passes periapse. The resulting phase plane plot is called a surface of section and consists of discrete points instead of continuous curves.

For the application to Mercury, the values of $\dot{\theta}$ and θ at periapse fall on smooth curves indistinguishable from those obtained from the approximate equation. The integrals of the motion, although analytically obscure, still appear to exist. We can now experiment with the surface of section by increasing $(B-A)/C$ in (1). The discrete points $(\theta, \dot{\theta})$ at periapse are still confined to a closed curve with $\langle\dot{\theta}\rangle = pn$ when initial conditions are near one of the resonances, and to a smooth curve of circulation when initial conditions are far from resonance. However, initial conditions near a separatrix lead to points $(\theta, \dot{\theta})$ confined to a definite region surrounding the separatrix but randomly distributed over that region. The motion has become chaotic in the sense that infinitesimal changes in initial conditions lead to drastically different trajectories in phase, and the definite region around the separatrix over which the points $(\theta, \dot{\theta})$ are scattered is called a chaotic zone in the surface of section (Chirikov 1979). This is illustrated in figure 2 for $\omega_0 = 0.2n$ and $e = 0.1$.

Another consequence of increasing $(B-A)/C$ is that the widths of the resonances are increased. In figure 2 this corresponds to a spreading of the separatrix curves as the chaotic

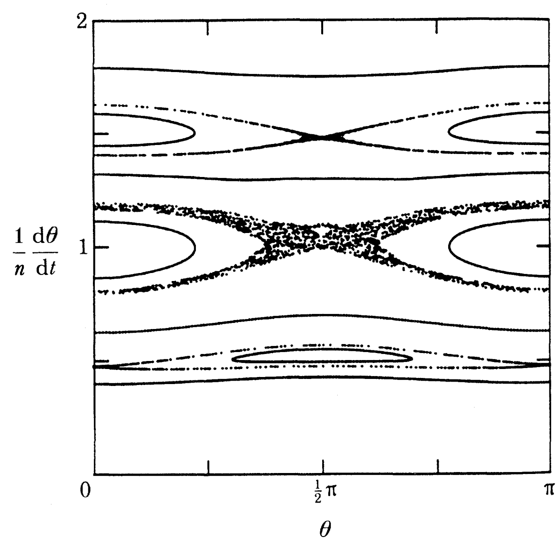

FIGURE 2. Surface of section for $\omega_0 = 0.2n$ and $e = 0.1$, illustrating that $d\theta/dt$ against θ at periapse passage has points falling on smooth curves for quasi-periodic libration or rotation, but that initial conditions near a separatrix lead to chaotic motion with successive points distributed uniformly over a chaotic zone. (After Wisdom *et al.* 1984.)

zone about each separatrix also increases in area. Continued increases in $(B-A)/C$ leads eventually to an overlap of nearby resonances in the sense that the separatrices of two adjacent resonances would now be close to each other at $\theta = 0$ or π if each resonance were treated as if no other resonances existed. In this case a widespread chaotic zone fills the region between the two resonances and surrounds islands of stable libration that shrink in area as $(B-A)/C$ is further increased. In a physical sense, initial values of $(\theta, \dot{\theta})$ could be consistent with libration about *either* of two adjacent resonant angular velocities if each of the resonances were

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respectively isolated. However, one resonance perturbs the other so strongly that libration in either resonance is impossible for those initial conditions and widespread chaotic motion follows.

Chirikov (1979) has proposed a resonance overlap criterion that states that when the sum of two unperturbed halfwidths equals the separation between resonance centres, large scale chaos ensues. For the $p = 1$ and $p = \frac{3}{2}$ resonances this criterion becomes

$$\omega_0 \sqrt{H(1,e)} + \omega_0 \sqrt{H(\tfrac{3}{2},e)} = \tfrac{1}{2}n, \tag{4}$$

with $H(1,e) \approx 1$ and $H(\frac{3}{2},e) \approx \frac{7}{2}e$, $\omega_0 \approx n[2+(14e)^{\frac{1}{2}}]^{-1}$. The chaotic region that exists around the separatrix for values of ω_0 considerably less than that necessary for resonance overlap in (4) can also be inferred to occur because of the overlap of other resonances. The period P of the motion becomes very large as a trajectory gets very near the separatrix and the corresponding frequency $2\pi/P$ arbitrarily small. If there is a nearby resonance with frequency ω_1, we see that as we ease up to the separatrix, there are an infinity of resonances with $2\pi N/P = \omega_1$ where N is an integer. Each resonance has a width, and the extent of the chaotic zone around each separatrix is estimated by how near the trajectory must be to the separatrix (how small $2\pi/P$ must be) for two of these resonances to overlap (Chirikov 1979).

For Hyperion, the mean value of e is 0.1 and the value of ω_0 determined from Voyager 2 images is $\omega_0 = (0.89 \pm 0.22)\,n$ (Duxbury, personal communication, 1983). But the value of ω_0 for widespread chaotic motion near $\dot{\theta} = n$ and $\frac{3}{2}n$ from (4) is $0.31n$ for $e = 0.1$, which implies that for Hyperion there is a large chaotic zone surrounding these resonances and probably more. Figure 3 shows the surface of section for Hyperion with $\omega_0 = 0.89n$, $e = 0.1$ verifying the large chaotic zone which in fact engulfs all states from $p = \frac{1}{2}$ to $p = \frac{9}{4}$, the latter being a second-order resonance.

The island of quasi-periodic librations for $p = \frac{1}{2}$ is reduced to the small remnant in the lower centre of the sea of chaos ($H(\frac{1}{2},e) < 0$, so the satellite librates about $\theta = \frac{1}{2}\pi$ for $p = \frac{1}{2}$) and the

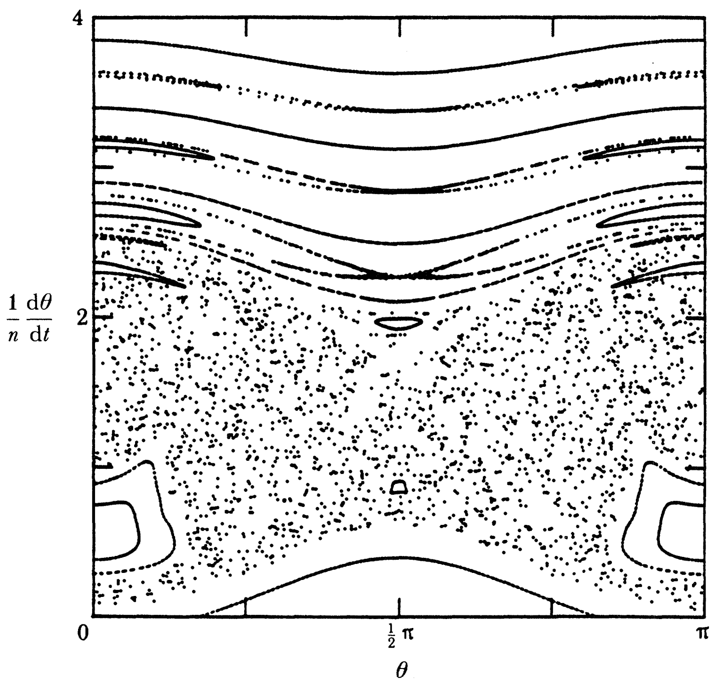

FIGURE 3. Surface of section appropriate for Hyperion ($\omega_0 = 0.89n$, $e = 0.1$). A widespread chaotic zone surrounds islands of librational stability at all spin–orbit states from $p = \frac{1}{2}$ to $p = \frac{9}{4}$. There is no stable libration about the $p = \frac{3}{2}$ state. (After Wisdom *et al.* 1984.)

$\frac{3}{2}$ state has disappeared altogether. This is understood since the halfwidth of the unperturbed $p = 1$ state is larger than the separation of the $p = 1$ and $p = \frac{3}{2}$ resonances. Quasi-periodic libration is also shown by the points falling on smooth curves for $p = 1, 2, \frac{9}{4}, \frac{5}{2}, 3$ and $\frac{7}{2}$ states. The centres of these islands of stable libration are substantially displaced from the mean values $\langle\dot{\theta}\rangle \equiv pn$ on the surface of section because of the large forced libration with the period of the orbit.

3. Stability of axis orientation

If Hyperion's spin axis were to remain normal to its orbit plane, tidal dissipation would most likely bring the spin to the chaotic zone, since capture into any of the higher-order resonances has negligibly small probability (see Goldreich & Peale 1966). Once within the chaotic zone, all points within it are repeatedly accessible and the rotation could be trapped into any of the resonances represented by the islands of quasi-periodic motion in the chaotic sea in figure 3. Old ideas about capture probabilities (Goldreich & Peale 1966) have to be revised, however, since now the satellite would have many chances at each accessible resonance instead of just one, as in the Mercury case. In fact, even in the situation where the chaotic region is limited to a narrow region about the separatrix, the capture probabilities would remain as before only if the tides could drag the spin across the width of the chaotic zone in a time less than the cycle time of γ (Wisdom *et al.* 1984). If Hyperion were to be trapped into an island, continuing dissipation would drive the trajectory to the island centre representing the forced periodic libration. We might have therefore expected to find Hyperion librating near the centre of the synchronous island in figure 3.

But this last expectation is only possible if the spin axis remains near the orbit normal during the periodic motion represented by an island centre in figure 3. In fact, Floquet theory (see, for example, Kane 1965) applied to the Euler equations shows the axis orientation normal to the orbit plane to be unstable at the $p = 1$ and $p = \frac{1}{2}$ states for nearly all the values of ω_0, within the uncertainties of the observations, but stable at the $p = 2$ and $\frac{9}{4}$ states (Wisdom *et al.* 1984). The Euler equations for rigid body motion (see, for example, Goldstein 1980, p. 158) are solved in terms of angles, like the Euler angles to specify the body orientation. Inertial axes are fixed in the orbit with the x-axis directed from Saturn to Hyperion's orbit periapse, the y-axis in the direction of orbital motion at periapse, and the z-axis parallel to the orbital angular momentum. If a, b, c are principal axes of the satellite which start coincident with the xyz-axes, θ is a rotation about c, ϕ a rotation about the new position of a, and ψ is a rotation about the last position of b instead of a rotation about c to avoid a coordinate singularity at $\phi = 0$ which is the point about which we wish to test the stability. The notation for θ and ϕ is reversed from that usually seen, to coincide with our earlier definition of θ as rotation about the spin axis when the axis is normal to the orbit plane.

With the components of the angular velocity along the body axes, and the direction cosines of the vector to Saturn relative to these axes expressed in terms of θ, ϕ, ψ, $\dot{\theta}$, $\dot{\phi}$, $\dot{\psi}$, we can write the Euler equations in terms of these variables, solve for $\ddot{\theta}$, $\ddot{\phi}$, $\ddot{\psi}$ in terms of angles and angular velocities, and set down the six equations to be solved as

$$\left.\begin{aligned} \mathrm{d}\dot{\theta}/\mathrm{d}t &= f_1(\theta,\phi,\psi,\dot{\theta},\dot{\phi},\dot{\psi}), & \mathrm{d}\theta/\mathrm{d}t &= \dot{\theta},\\ \mathrm{d}\dot{\phi}/\mathrm{d}t &= f_2(\theta,\phi,\psi,\dot{\theta},\dot{\phi},\dot{\psi}), & \mathrm{d}\phi/\mathrm{d}t &= \dot{\phi},\\ \mathrm{d}\dot{\psi}/\mathrm{d}t &= f_3(\theta,\phi,\psi,\dot{\theta},\dot{\phi},\dot{\psi}), & \mathrm{d}\psi/\mathrm{d}t &= \dot{\psi}. \end{aligned}\right\} \tag{5}$$

As the motion is periodic at the island centres, the coefficients in these equations are periodic. This satisfies the necessary condition for the application of Floquet theory. If p_θ, p_ϕ, p_ψ are momenta conjugate to θ, ϕ, ψ, then $\phi, \psi, p_\phi, p_\psi$ are zero if the axis is started perpendicular to the orbit plane and they remain zero if the axis is undisturbed. In this case θ is identical to the θ used earlier in this section. A trajectory near the periodic trajectory is specified by $\theta' = \theta + \delta\theta$, $\phi' = \phi + \delta\phi$, $\psi' = \psi + \delta\psi$, $p'_\theta = p_\theta + \delta p'_\theta$, $p'_\phi = p_\phi + \delta p_\phi$, $p'_\psi = p_\psi + \delta p_\psi$. A linear transformation between the values of the variables at $t = 0$ and $t = \tau$ (one period later) is generated by the following: assume all the initial increments in the canonical variables are zero except one, which is given a small value; find the corresponding initial values of $(\dot{\theta}, \dot{\phi}, \dot{\psi}, \theta, \phi, \psi)$, numerically integrate (5) over one period to find the corresponding increments $\delta p_\theta(\tau)$, $\delta p_\phi(\tau)$, $\delta p_\psi(\tau)$, $\delta\theta(\tau)$, $\delta\phi(\tau)$, $\delta\psi(\tau)$; and repeat this process by assuming a different variable to be incremented initially. This leads to a new set of increments for all the variables one period later. The procedure ends when all variables have been so treated. This gives six sets of increments at $t = \tau$ corresponding to the six independent initial conditions which deviate slightly from the periodic solution. Normalize each set by dividing each member of the set by the initial increment which generated it. If we form a 6×6 matrix with the normalized sets of increments as columns, the matrix represents a linear transformation which maps an arbitrary initial set of increments into their values one period later. As the coefficients in the equations have returned to their original values, we may apply the same linear transformation repeatedly to step the solution a period at a time. If any of the eigenvalues of this transformation (called Floquet multipliers) have a modulus greater than one, the periodic motion is linearly unstable. For canonical variables in a Hamiltonian system, each eigenvalue with modulus greater than one is matched by another with modulus less than one such that the product of all the eigenvalues is one (Poincaré, 1892). A necessary condition for stability is thus that all the eigenvalues of the transformation matrix have modulus one. The imaginary exponents of the eigenvalues yield the characteristic frequencies of the perturbed motion.

Floquet theory only tests for the linear stability of the axis orientation. To check that the spin axis is not stabilized slightly displaced from a direction normal to the orbit plane by nonlinear effects, or that it exhibits a large amplitude, periodic variation, trajectories starting near the equilibrium orientation were integrated numerically. In every case, the spin axis and c-body axis (not remaining coincident with the spin) went more than 90° from their original orientations perpendicular to the orbit plane. There is no nonlinear stabilization and the motion is not periodic. This instability precludes the stable libration of Hyperion in the $p = 1$ or $\frac{1}{2}$ states, which was possible as long as we found the spin axis perpendicular to the orbit plane. Libration in the $p = 2$ or $\frac{9}{4}$ states is still possible but the behaviour of the spin axis throughout the chaotic zone makes capture into either of these states improbable.

As motion in the chaotic zone is not periodic, Floquet theory cannot be used to test the stability of the axis orientation. However, in the chaotic zone we can numerically integrate one trajectory corresponding to the spin axis being normal to the orbit plane, while simultaneously integrating a second trajectory that starts with spin axis only slightly displaced. If the spin axes for such neighbouring trajectories separate from the equilibrium orientation exponentially, the spin axis in the reference trajectory is attitude unstable. A measure of the exponential separation of two nearby trajectories are the Lyapunov characteristic exponents (see, for example, Wisdom 1983) defined by

$$\lambda = \lim_{t \to \infty} [\ln (d(t)/d(t_0))/(t - t_0)], \tag{6}$$

where

$$d(t) = [\delta\theta^2 + \delta\phi^2 + \delta\psi^2 + \delta p_\theta^2 + \delta p_\phi + \delta p_\psi^2]^{\frac{1}{2}}$$

is the ordinary Euclidean separation between two nearby trajectories. The conjugate variables obey the same equations as those derived from the Euler equations, but the motion need not be periodic. For the Hamiltonian system considered here there are three independent non-negative Lyapunov exponents corresponding to the three degrees of freedom, and the routine integration of two nearby trajectories in (6) leads naturally to the largest one (Benettin *et al.* 1980*a*). But in the chaotic zone, one of these positive exponents is already associated with the chaotic rotation about the spin axis fixed normal to the orbit plane. Instability of the axis orientation requires that at least two of the independent exponents be positive. A numerical algorithm developed by Benettin *et al.* (1980*b*) is used to find all three independent exponents with the result that the axis orientation is unstable throughout the chaotic zone shown in figure 3.

A positive Lyapunov characteristic exponent for a given reference trajectory is a signature of chaotic motion. For trajectories whose initial conditions are slightly displaced from the equilibrium orientation of the spin axis in the chaotic zone all three exponents are positive. So in addition to the attitude instability, the rotation of Hyperion is chaotic for all degrees of freedom in this zone and it must tumble in an essentially random manner with spin magnitude and axis orientation undergoing large variations on timescales comparable with an orbit period. A similar determination of the λ for two close trajectories starting near the periodic solution show that the motion of the spin axis away from the orbit normal direction is also fully chaotic for the $p = 1$ and $p = \frac{1}{2}$ states.

4. Discussion

For values of the spin magnitude $\dot{\theta} \gtrsim 3n$, Hyperion would be expected to experience a conventional tidal evolution where the spin magnitude is retarded and the obliquity (angle between the spin axis and the orbit normal) tends toward an equilibrium value which itself goes to zero as $\dot{\theta} \rightarrow 2n$ (Goldreich & Peale 1970). After Hyperion enters the chaotic zone with its obliquity near zero, it is forced to tumble in the manner described above. Capture into the synchronous and $p = \frac{1}{2}$ islands of libration is not possible since the spin axis is attitude unstable in these states. Capture into the stable libration states for $p = 2$ or $\frac{9}{4}$ is highly improbable as Hyperion must approach the islands with its spin axis close to the orbit normal and remain close to the island for a sufficiently long time for a weak tidal dissipation to capture it from a strongly chaotic region ($\lambda = 0.1$).

Although the chaotic tumbling has not been observationally verified, preliminary observations of a 13-day period (Goguen 1983; Thomas *et al.* 1984) support the conclusion that Hyperion has not been captured in one of the stable islands. Hyperion is expected to be found tumbling chaotically in which case it would represent the only example of confined, continuously observable chaotic motion in the Solar System (Wisdom *et al.* 1984).

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Discussion

L. M. B. C. Campos (*Department of Applied Mathematics and Theoretical Physics, Silver Street, Cambridge, U.K.*). Dr Peale's presentation was concluded by a computer film simulating the rotation of Hyperion, and based on a numerical integration of the three-dimensional nonlinear equations of motion, showing a 'random' tumbling.

The theory he had presented before concerned a one-dimensional, nonlinear system analogous to a pendulum, and one feels that there is a gap between the two cases of nearly symmetric and very unsymmetric tops. Did he do any theoretical analysis of the three-dimensional nonlinear system, and did it exhibit bifurcations, period doubling or other properties usually associated with the onset of chaotic motion, say, as the moments of inertia are taken to be increasingly dissimilar?

S. J. Peale. The nearly symmetric satellite was used to review the concept of resonances between spin and orbital motions and the pendulum-like stability of these resonances for eccentric orbits. The gap between this simple beginning and the film you saw was bridged in two steps: first, the averaging procedure leading to the pendulum equation that describes each resonance was shown to fail if the satellite was too asymmetric. One is then forced to use the complete equation of motion in one dimension but lack of an analytic solution makes numerical integration necessary. Widespread chaotic motion in the one-dimensional problem results when the satellite is so asymmetric that two adjacent resonances overlap in the sense I described, where 'widespread' refers to the area in the surface of section ($\dot{\theta}$ against θ at periapse passages). Hyperion is so deformed that several adjacent resonances overlapped and a chaotic sea envelops islands of quasi-periodic libration from the $\frac{1}{2}$ to the $\frac{9}{4}$ spin states. The second step to the film was to show that the axis orientation within the $p = 1$ and $p = \frac{1}{2}$ libration islands and throughout the chaotic zone on the surface of section is unstable. The linear stability analysis with the full three degrees of freedom was supplemented by the determination of the exponential separation of nearby trajectories in the six-dimensional phase space where the trajectories start at points corresponding to the spin axis being nearly perpendicular to the orbit plane. This exponential separation of trajectories is characteristic of chaotic motion, and indicates that the tumbling is not large amplitude periodic. The film is thus simply a display of the tumbling motion of Hyperion, and the assertion that the tumbling is fully chaotic was not deduced from the numerical integration which produced the film, but from the exponential separation of nearby trajectories.

In answer to the second question, distinct regions of chaotic and quasi-periodic trajectories in the phase space always exist where the former shrink to insignificance as the shape of the satellite becomes more spherical or as orbital eccentricity goes to zero. The route to chaos is by resonance overlap and not period doubling. The most important thing is that a large chaotic region exists in the six-dimensional phase space of Hyperion's rotation.

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